EVALUATION OF FIXED POINT QUASICONVEX SUBGRADIENT METHOD WITH COMPUTATIONAL INEXACTNESS

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Abstract. This paper considers constrained quasiconvex optimization problems and discusses the convergence of the fixed point quasiconvex subgradient method when errors and noise appear in the computation. For this aim, we propose an extension of the fixed point quasiconvex subgradient method that takes into account computational inexactness. The main theorem presented in this paper extends the range of the existing theorem on the exact fixed point quasiconvex subgradient method to cases with inexact parameters.

1. Introduction

This paper considers constrained quasiconvex optimization problems, in particular, for when the constraint set is expressed as the fixed point set of some nonexpansive mapping.

One of the most important instances of quasiconvex objective functionals is the fractional functional [5, 6, 11]. This functional is expressed as a fractional of two functionals and is used for modeling ratio indicators, such as the debt/equity in financial and corporate planning, inventory/sales and output/employee in production planning, and cost/patient and nurse/patient ratios in healthcare and hospital planning [11]. Solving problems in these applications is the main motivation behind this paper.

Let us survey the existing studies. Kiwiel proposed the quasiconvex subgradient method for solving constrained quasiconvex optimization problems [7]. This method uses a subgradient, which is defined as a normalized normal vector to the slice for optimizing the quasiconvex functional. Fortunately, this subgradient can be easily obtained when the objective functional is a fractional one [7, Lemma 4]. Hence, this method is useful for solving constrained quasiconvex or fractional optimization problems. Hu proposed the inexact quasiconvex subgradient method, which includes consideration of sources of inexactness such as computation errors and noise that come from practical considerations and applications [5]. The inexact quasiconvex

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subgradient method and its detailed convergence analysis indicate how accurate the solution obtained from the quasiconvex subgradient method will be even if there is some inexactness in the computation. The quasiconvex subgradient method can be applied to constrained quasiconvex optimization problems. However, it uses the metric projection onto the constraint set for letting the solution be contained in it. This implies that the metric projection onto the constraint set must be computed in order to use the quasiconvex subgradient method.

The fixed point quasiconvex subgradient method has been proposed to relieve the assumption of computability of the metric projection onto the constraint set. This method combines the Krasnosel'skiĭ-Mann algorithm [8, 9] with the quasiconvex subgradient method. A nonexpansive mapping is an extension of the metric projection, and its fixed point set can express a wider class of constraint sets than the metric projection can. The Krasnosel'skiĭ-Mann algorithm [8, 9] is one that finds a fixed point of a given nonexpansive mapping. Thus, the fixed point quasiconvex subgradient method can optimize a quasiconvex objective functional over the fixed point set of a nonexpansive mapping and can solve a wider class of constrained quasiconvex optimization problems than the quasiconvex subgradient method can.

This paper proposes a fixed point quasiconvex subgradient method that takes into account computational inexactness and discusses its convergence property. Three kinds of inexactness are considered in this paper. The existing study on the inexact quasiconvex subgradient method [5] considers computation errors and noise appearing in the subgradient computation. In addition to these, this paper also considers noise appearing in the computation of the nonexpansive mapping, since the fixed point quasiconvex subgradient method uses a nonexpansive mapping instead of the metric projection used in the quasiconvex subgradient method.

The main theorem presented in this paper describes (i) how much errors affect the solution obtained by the proposed algorithm, (ii) what factors cause these errors, and (iii) how to bring down the error to under the desired tolerance. The main theorem is an extension of the existing theorem [4] on the exact fixed point quasiconvex subgradient method. Hence, this paper offers a more detailed analysis of the fixed point quasiconvex subgradient method.

This paper is organized as follows. Section 2 gives the mathematical preliminaries. Section 3 analyses the convergence of the inexact fixed point quasiconvex subgradient method. Section 4 concludes this paper.

2. Mathematical Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ and its induced norm $\|\cdot\| : H \to \mathbb{R}$. $\mathbb{N}$ is the set of natural numbers without zero, and $\mathbb{R}$ is the set of real numbers. A functional $f : H \to \mathbb{R}$ is called *quasiconvex* if $f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$ for every $x, y \in H$ and $\alpha \in [0,1]$ [1, Definition 5.1], [3, Definition (4.4)], [10, Chapter 2]. The effective domain of a functional $f : H \to \mathbb{R}$ is defined as $\mathrm{dom}(f) := \{x \in H : f(x) < \infty\}$.

Here, let $f : H \to \mathbb{R}$ be a quasiconvex, continuous functional and let $X \subset H$ be a nonempty, closed, convex set. Then, the main problem of this paper is to

$$\text{minimize } f(x) \text{ subject to } x \in X. \tag{2.1}$$

We define the *set of minima* and the *minimum value* of Problem (2.1) by $X^\star := \mathrm{argmin}_{x \in X} f(x)$ and $f_\star := \inf_{x \in X} f(x)$, respectively.

Let us define the other terms and notations which will be used in the later discussion. $\mathbf{B} := \{x \in H : \|x\| \leq 1\}$ is the unit ball in this Hilbert space, and $\mathbf{S} := \{x \in H : \|x\| = 1\}$ is the unit sphere in that space. Id is the identity mapping of H onto itself, and the closure of a set $C \subset H$ is denoted by $\mathrm{cl}\, C$.

The metric projection onto a closed, convex set $C \subset H$, denoted by P_C, is defined as $P_C(x) \in C$ such that $\|x - P_C(x)\| = \inf_{y \in C} \|x - y\|$ for any $x \in H$. For any $\alpha \in \mathbb{R}$, the α-slice of a functional $f : H \to \mathbb{R}$ is defined as $\mathrm{lev}_{<\alpha} f := \{x \in H : f(x) < \alpha\}$. A mapping $T : H \to H$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for any $x, y \in H$, and it is said to be firmly nonexpansive if $\|T(x) - T(y)\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2$ for any $x, y \in H$. Obviously, a firmly nonexpansive mapping is also a nonexpansive mapping [2, Subchapter 4.1]. The properties of these nonexpansivities are described in detail in [2, Chapter 4], [12, Chapter 6]. The fixed point set of a mapping $T : H \to H$ is defined as $\mathrm{Fix}(T) := \{x \in H : T(x) = x\}$.

The distance of a vector $x \in H$ from a set $Z \subset H$ is defined as $\mathrm{dist}(x, Z) := \inf_{z \in Z} \|x - z\|$ [5, Subsection 2.1]. A functional $f : H \to \mathbb{R}$ is said to satisfy the *Hölder condition* of order $p > 0$ with modulus $\mu > 0$ on H if $f(x) - f_\star \leq \mu(\mathrm{dist}(x, X^\star))^p$ holds for all $x \in H$ [5, Assumption 2]. For given a point $x \in H$ and for a nonnegative real $\epsilon \geq 0$, we call the set $\bar{\partial}^\star_\epsilon f(x) := \{g \in H : \langle g, y - x \rangle \leq 0 \ (y \in \mathrm{lev}_{<f(x)-\epsilon} f)\}$ the ϵ-subdifferential of the quasiconvex functional f at a point $x \in H$ [5, Definition 2.4]. We also call any of its element a *subgradient.*

We propose Algorithm 1 for considering the effect of computational inexactness on the fixed point quasiconvex subgradient method. The difference from the original fixed point quasiconvex subgradient method [4, Algorithm 1] is the three sequences $\{\epsilon_k\}$, $\{r^f_k\}$, and $\{r^T_k\}$. The sequences $\{\epsilon_k\}$ and $\{r^f_k\}$ are from [5], and they express the computational errors and noise, respectively. In addition to these sequences, we consider $\{r^T_k\}$, which expresses the noise appearing in the computation of the nonexpansive mapping. If these sequences are always zero, Algorithm 1 coincides with the existing fixed point quasiconvex subgradient method [4, Algorithm 1].

Algorithm 1 Fixed point quasiconvex subgradient method [4] with inexactness

Require:
$f\colon H \to \mathbb{R}$, $T\colon H \to H$.
$\{v_k\} \subset (0, \infty)$, $\{\alpha_k\} \subset (0, 1]$. ▷ Hyperparameters
$\{\epsilon_k\} \subset [0, \infty)$, $\{r_k^f\} \subset H$, $\{r_k^T\} \subset H$. ▷ Inexactness

Ensure:
This algorithm generates a sequence $\{x_k\} \subset H$.

1: $x_1 \in H$.
2: **for** $k = 1, 2, \ldots$ **do**
3: $g_k \in \bar{\partial}^{\star}_{\epsilon_k} f(x_k) \cap \mathbf{S}$.
4: $\tilde{g}_k := g_k + r_k^f$, $\tilde{T}_k := T + r_k^T$.
5: $x_{k+1} := \alpha_k x_k + (1 - \alpha_k)\tilde{T}_k(x_k - v_k \tilde{g}_k)$.
6: **end for**

The following assumption and propositions will be used in the later discussion.

Assumption 2.1. We suppose that

(A1) the effective domain $\mathrm{dom}(f) := \{x \in H : f(x) < \infty\}$ coincides with the whole space H;
(A2) there exists some firmly nonexpansive mapping $T : H \to H$ whose fixed point set $\mathrm{Fix}(T)$ coincides with the constraint set X;
(A3) the constraint set X is nonempty, and there exists at least one minimum, i.e. $X^\star \neq \emptyset$;
(A4) the generated sequence $\{x_k\}$ is bounded [5, Assumption 1];
(A5) the functional f satisfies the Hölder condition of order $p > 0$ with modulus $\mu > 0$ on H [5, Assumption 2];
(A6) the sequence $\{\alpha_k\} \subset (0, 1]$ satisfies $0 < \liminf_{k\to\infty} \alpha_k \leq \limsup_{k\to\infty} \alpha_k < 1$ [4, Assumption 3.1];
(A7) there exist some $R_f, R_T, \epsilon \geq 0$ such that $\left\| r_k^f \right\| \leq R_f$ for all $k \in \mathbb{N}$, $\limsup_{k\to\infty} \left\| r_k^T \right\| = R_T$, and $\limsup_{k\to\infty} \epsilon_k = \epsilon$ [5, Assumption 3];
(A8) the sequence $\{v_k\} \subset (0, \infty)$ converges to some nonnegative real $v \in [0, \infty)$, $\sum_{k=1}^{\infty} v_k = \infty$, and there exists a nonnegative real $c \geq 0$ such that $\left\| r_k^T \right\| \leq c v_k$ for all $k \in \mathbb{N}$.

Proposition 2.2 ([7, Lemma 6.(b)])**.** *If* $\bar{x} + \bar{r}\mathbf{B} \subset \mathrm{cl}\left(\mathrm{lev}_{<f(x_k)-\epsilon_k} f\right)$ *for some* $\bar{x} \in H$ *and* $\bar{r} \geq 0$*, then* $\langle g_k, x_k - \bar{x}\rangle \geq \bar{r}$ *holds.*

Proposition 2.3 ([5, Lemma 3.3])**.** *Let* $\{x_k\}$ *be the sequence generated by Algorithm 1, and suppose that Assumption 2.1 holds. If* $f(x_k) > f_\star + \mu\bar{r}^p + \epsilon_k$ *holds for some* $\bar{r} \geq 0$*, then* $\langle g_k, x_k - x^\star\rangle \geq \bar{r}$ *for all* $x^\star \in X^\star$*.*

Proof. Fix $x^\star \in X^\star$ arbitrarily. Assumption (A5) guarantees that the Hölder condition of order p with modulus μ holds for the point x_k. Fix $x \in x^\star + \bar{r}\mathbf{B}$

arbitrarily. Then, $\|x - x^\star\| \leq \bar{r}$ holds. Hence, together with the assumption of this proposition, we have

$$f(x) - f_\star \leq \mu \left(\operatorname{dist}(x, X^\star)\right)^p \leq \mu \bar{r}^p < f(x_k) - f_\star - \epsilon_k.$$

The above inequality implies that $x \in \operatorname{lev}_{f(x_k)-\epsilon_k} f$, and thus $x^\star + \bar{r}\mathbf{B} \subset \operatorname{lev}_{<f(x_k)-\epsilon_k} f$. The result follows from Proposition 2.2. □

3. Convergence Analysis

First, let us show the two fundamental inequalities for evaluating the objective functional value and the degree of approximation to the fixed point set.

Lemma 3.1. *Let $\{x_k\}$ be the sequence generated by Algorithm 1, and let Assumption 2.1 hold. Suppose that $x^\star \in X^\star$. Then, there exists a constant $M_1 \geq 0$ such that*

$$\begin{aligned}
&\|x_{k+1} - x^\star\|^2 \\
&\leq \|x_k - x^\star\|^2 - (1-\alpha_k)\Bigg(2v_k\bigg(\langle g_k, x_k - x^\star\rangle - R_f M_1 - \frac{1}{2}v_k(R_f+1)^2 \\
&\quad - \left\|r_k^T\right\|(R_f+1)\bigg) - \left\|r_k^T\right\|\left(\left\|r_k^T\right\| + 2M_1\right)\Bigg).
\end{aligned}$$

for all $k \in \mathbb{N}$.

Proof. Fix $x^\star \in X^\star$ and $k \in \mathbb{N}$ arbitrarily. The convexity of $\|\cdot\|^2$ ensures that

$$\begin{aligned}
\|x_{k+1} - x^\star\|^2 &= \left\|\alpha_k x_k + (1-\alpha_k)\tilde{T}_k(x_k - v_k\tilde{g}_k) - x^\star\right\|^2 \\
&\leq \alpha_k \|x_k - x^\star\|^2 + (1-\alpha_k)\left\|\tilde{T}_k(x_k - v_k\tilde{g}_k) - x^\star\right\|^2. \qquad (3.1)
\end{aligned}$$

Here, let us consider the right term on the right side of the above inequality. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left\|\tilde{T}_k(x_k - v_k\tilde{g}_k) - x^\star\right\|^2 \\
&= \left\|T(x_k - v_k\tilde{g}_k) - x^\star + r_k^T\right\|^2 \\
&= \|T(x_k - v_k\tilde{g}_k) - x^\star\|^2 + \left\|r_k^T\right\|^2 + 2\left\langle r_k^T, T(x_k - v_k\tilde{g}_k) - x^\star\right\rangle \\
&\leq \|T(x_k - v_k\tilde{g}_k) - x^\star\|^2 + \left\|r_k^T\right\|^2 + 2\left\|r_k^T\right\|\|T(x_k - v_k\tilde{g}_k) - x^\star\|.
\end{aligned}$$

Since the point $x^\star$ is a fixed point of the nonexpansive mapping T, the nonexpansivity of T ensures that

$$\begin{aligned}
&\left\|\tilde{T}_k(x_k - v_k\tilde{g}_k) - x^\star\right\|^2 \\
&\leq \|x_k - v_k\tilde{g}_k - x^\star\|^2 + \left\|r_k^T\right\|^2 + 2\left\|r_k^T\right\|\|x_k - v_k\tilde{g}_k - x^\star\| \\
&\leq \|x_k - v_k\tilde{g}_k - x^\star\|^2 + 2v_k\left\|r_k^T\right\|\|\tilde{g}_k\| + \left\|r_k^T\right\|\left(\left\|r_k^T\right\| + 2\|x_k - x^\star\|\right).
\end{aligned}$$

Assumption (A7) guarantees that the noise vectors r_k^f are bounded from above, i.e., $\|r_k^f\| \leq R_f$, and this implies that $\|\tilde{g}_k\| \leq \|g_k\| + \|r_k^f\| \leq R_f + 1$. Assumption (A4) implies that there exists a constant $M \geq 0$ such that $\|x_k\| \leq M$ for all $k \in \mathbb{N}$, and the sequence $\{\|x_k - x^\star\|\}$ is also bounded because $\|x_k - x^\star\| \leq \|x_k\| + \|x^\star\| \leq M + \|x^\star\|$ for all $k \in \mathbb{N}$. Let us define the constant M_1 as this upper bound. Hence, we obtain

$$\begin{aligned} &\left\|\tilde{T}_k(x_k - v_k\tilde{g}_k) - x^\star\right\|^2 \\ &\leq \|x_k - v_k\tilde{g}_k - x^\star\|^2 + 2v_k \left\|r_k^T\right\| (R_f + 1) + \left\|r_k^T\right\| \left(\left\|r_k^T\right\| + 2M_1\right) \end{aligned} \tag{3.2}$$

from the above inequality. Next, let us evaluate the left-most term on the right side of the above inequality. Expanding the term and rearranging it, we have

$$\begin{aligned} &\|x_k - v_k\tilde{g}_k - x^\star\|^2 \\ &= \|x_k - x^\star\|^2 - 2v_k \langle \tilde{g}_k, x_k - x^\star\rangle + v_k^2 \|\tilde{g}_k\|^2 \\ &= \|x_k - x^\star\|^2 - 2v_k \left(\langle g_k, x_k - x^\star\rangle + \left\langle r_k^f, x_k - x^\star \right\rangle - \frac{1}{2} v_k \|\tilde{g}_k\|^2 \right). \end{aligned}$$

Using the Cauchy-Schwarz inequality, the term $\langle r_k^f, x_k - x^\star\rangle$ can be bounded from above by $\langle r_k^f, x_k - x^\star\rangle \leq \|r_k^f\| \, \|x_k - x^\star\| \leq R_f M_1$. Hence, we obtain

$$\begin{aligned} &\|x_k - v_k\tilde{g}_k - x^\star\|^2 \\ &\leq \|x_k - x^\star\|^2 - 2v_k \left(\langle g_k, x_k - x^\star\rangle - R_f M_1 - \frac{1}{2} v_k (R_f + 1)^2 \right). \end{aligned} \tag{3.3}$$

The obtained inequalities (3.1), (3.2) and (3.3) imply that

$$\begin{aligned} &\|x_{k+1} - x^\star\|^2 \\ &\leq \|x_k - x^\star\|^2 - (1 - \alpha_k)\Bigg(2v_k\Bigg(\langle g_k, x_k - x^\star\rangle - R_f M_1 - \frac{1}{2} v_k (R_f + 1)^2 \\ &\quad - \left\|r_k^T\right\| (R_f + 1)\Bigg) - \left\|r_k^T\right\| \left(\left\|r_k^T\right\| + 2M_1\right) \Bigg). \end{aligned}$$

This completes the proof. □

Lemma 3.2. *Let $\{x_k\}$ be the sequence generated by Algorithm 1, and let Assumption 2.1 hold. Suppose that $z \in \mathrm{Fix}(T)$. Then, a constant $M_2 \geq 0$ exists such that*

$$\begin{aligned} \|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 &- (1 - \alpha_k) \|x_k - T(x_k - v_k\tilde{g}_k)\|^2 \\ &+ 2v_k(R_f + 1)M_2 + 2\left\|r_k^T\right\| M_2 + \left\|r_k^T\right\|^2. \end{aligned}$$

for all $k \in \mathbb{N}$.

Proof. Fix $z \in \mathrm{Fix}(T)$ and $k \in \mathbb{N}$ arbitrarily. Using the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{k+1} - z\|^2 &= \left\|\alpha_k x_k + (1-\alpha_k)\tilde{T}(x_k - v_k\tilde{g}_k) - z\right\|^2 \\ &\leq \alpha_k \|x_k - z\|^2 + (1-\alpha_k)\left\|\tilde{T}(x_k - v_k\tilde{g}_k) - z\right\|^2. \end{aligned} \tag{3.4}$$

Next, let us consider the term $\|\tilde{T}(x_k - v_k\tilde{g}_k) - z\|^2$. Expanding this term leads to

$$\begin{aligned}\left\|\tilde{T}(x_k - v_k\tilde{g}_k) - z\right\|^2 &= \left\|T(x_k - v_k\tilde{g}_k) - z + r_k^T\right\|^2 \\ &= \|T(x_k - v_k\tilde{g}_k) - z\|^2 + 2\langle r_k, T(x_k - v_k\tilde{g}_k) - z\rangle + \left\|r_k^T\right\|^2. \end{aligned} \tag{3.5}$$

Let us further consider the term $\|T(x_k - v_k\tilde{g}_k) - z\|^2$. With the firm nonexpansivity of T, an upper bound of the term can be estimated at

$$\begin{aligned}&\|T(x_k - v_k\tilde{g}_k) - z\|^2 \\ &\leq \|x_k - v_k\tilde{g}_k - z\|^2 - \|(\mathrm{Id} - T)(x_k - v_k\tilde{g}_k) - (\mathrm{Id} - T)(z)\|^2 \\ &= \|x_k - v_k\tilde{g}_k - z\|^2 - \|x_k - v_k\tilde{g}_k - T(x_k - v_k\tilde{g}_k)\|^2 \\ &= \|x_k - z\|^2 - 2\langle v_k\tilde{g}_k, x_k - z\rangle + v_k^2\|\tilde{g}_k\|^2 \\ &\quad - \Big(\|x_k - T(x_k - v_k\tilde{g}_k)\|^2 - 2\langle v_k\tilde{g}_k, x_k - T(x_k - v_k\tilde{g}_k)\rangle + v_k^2\|\tilde{g}_k\|^2\Big) \\ &= \|x_k - z\|^2 - \|x_k - T(x_k - v_k\tilde{g}_k)\|^2 + 2\langle v_k\tilde{g}_k, z - T(x_k - v_k\tilde{g}_k)\rangle. \end{aligned}$$

From equality (3.5), we obtain an upper bound of the term $\|\tilde{T}(x_k - v_k\tilde{g}_k) - z\|^2$ as follows

$$\begin{aligned}\left\|\tilde{T}(x_k - v_k\tilde{g}_k) - z\right\|^2 \leq \|x_k - z\|^2 - \|x_k - T(x_k - v_k\tilde{g}_k)\|^2 + \left\|r_k^T\right\|^2 \\ + 2\left\langle v_k\tilde{g}_k - r_k^T, z - T(x_k - v_k\tilde{g}_k)\right\rangle. \end{aligned}$$

Assumption (A4) implies that there exists a constant $M \geq 0$ such that $\|x_k\| \leq M$ for all $k \in \mathbb{N}$, and the sequence $\{\|z - T(x_k - v_k\tilde{g}_k)\|\}$ is also bounded because $\|z - T(x_k - v_k\tilde{g}_k)\| \leq \|z\| + \|x_k\| + v_k\|\tilde{g}_k\| \leq \|z\| + M + (\sup_{j\in\mathbb{N}} v_j)(R_f + 1) < \infty$ for all $k \in \mathbb{N}$. Let us define the constant M_2 as this upper bound. Together with the Cauchy-Schwarz inequality, the above inequality implies that

$$\begin{aligned}\left\|\tilde{T}(x_k - v_k\tilde{g}_k) - z\right\|^2 &\leq \|x_k - z\|^2 - \|x_k - T(x_k - v_k\tilde{g}_k)\|^2 \\ &\quad + 2(v_k\|\tilde{g}_k\| + \left\|r_k^T\right\|)M_2 + \left\|r_k^T\right\|^2 \\ &\leq \|x_k - z\|^2 - \|x_k - T(x_k - v_k\tilde{g}_k)\|^2 \\ &\quad + 2v_k(R_f + 1)M_2 + 2\left\|r_k^T\right\|M_2 + \left\|r_k^T\right\|^2. \end{aligned}$$

Combining this estimation with inequality (3.4), we obtain

$$\begin{aligned}\|x_{k+1}-z\|^2 &\le \alpha_k \|x_k - z\|^2 + (1-\alpha_k)\Big(\|x_k - z\|^2 - \|x_k - T(x_k - v_k\tilde{g}_k)\|^2 \\ &\qquad +2v_k(R_f+1)M_2 + 2\left\|r_k^T\right\| M_2 + \left\|r_k^T\right\|^2\Big) \\ &\le \|x_k - z\|^2 - (1-\alpha_k)\|x_k - T(x_k - v_k\tilde{g}_k)\|^2 \\ &\qquad + 2v_k(R_f+1)M_2 + 2\left\|r_k^T\right\| M_2 + \left\|r_k^T\right\|^2,\end{aligned}$$

and we have arrived at the desired inequality. □

The following lemma will be used for the proof by contradiction.

Lemma 3.3. *Let $\{x_k\}$ be the sequence generated by Algorithm 1, and let Assumption 2.1 hold. If a number $k \in \mathbb{N}$ and a nonnegative real $\delta \ge 0$ exist such that*

$$f(x_k) > f_\star + \mu\left((R_f+c)M_1 + \frac{1}{2}v_k(R_f+1)^2 + \left\|r_k^T\right\|\left(R_f+1+\frac{c}{2}\right)+\delta\right)^p + \epsilon_k,$$

then,

$$\|x_{k+1}-x^\star\|^2 \le \|x_k - x^\star\|^2 - 2v_k\delta(1-\alpha_k)$$

holds for any $x^\star \in X^\star$.

Proof. Fix $x^\star \in X^\star$ arbitrarily. Lemma 3.1 implies that

$$\begin{aligned}\|x_{k+1}-x^\star\|^2 &\le \|x_k - x^\star\|^2 - (1-\alpha_k)\bigg(2v_k\bigg(\langle g_k, x_k - x^\star\rangle - R_f M_1 \\ &\qquad - \frac{1}{2}v_k(R_f+1)^2 - \left\|r_k^T\right\|(R_f+1)\bigg) - \left\|r_k^T\right\|\left(\left\|r_k^T\right\| + 2M_1\right)\bigg).\end{aligned}$$

Assumption (A8) guarantees that $\left\|r_k^T\right\| < cv_k$. Hence, we obtain

$$\begin{aligned}\|x_{k+1}-x^\star\|^2 &\le \|x_k - x^\star\|^2 - (1-\alpha_k)\bigg(2v_k\bigg(\langle g_k, x_k - x^\star\rangle - R_f M_1 \\ &\qquad - \frac{1}{2}v_k(R_f+1)^2 - \left\|r_k^T\right\|(R_f+1)\bigg) - cv_k\left(\left\|r_k^T\right\| + 2M_1\right)\bigg) \\ &= \|x_k - x^\star\|^2 - 2v_k(1-\alpha_k)\bigg(\langle g_k, x_k - x^\star\rangle - (R_f+c)M_1 \\ &\qquad - \frac{1}{2}v_k(R_f+1)^2 - \left\|r_k^T\right\|\left(R_f+1+\frac{c}{2}\right)\bigg).\end{aligned}$$

Here, from Proposition 2.3 with the assumption of this lemma, we have

$$\langle g_k, x_k - x^\star\rangle \ge (R_f+c)M_1 + \frac{1}{2}v_k(R_f+1)^2 + \left\|r_k^T\right\|\left(R_f+1+\frac{c}{2}\right)+\delta.$$

Hence, the above two inequalities leads to the desired inequality

$$\|x_{k+1}-x^\star\|^2 \le \|x_k - x^\star\|^2 - 2v_k\delta(1-\alpha_k).$$

This completes the proof. □

Now, let us prove the following lemma, which will be used for proving the main theorem.

Lemma 3.4. *Let $\{x_k\}$ be the sequence generated by Algorithm 1, and let Assumption 2.1 hold. Then, there exists a subsequence $\{x_{k_i}\} \subset \{x_k\}$ which satisfies*

(i)
$$\lim_{i\to\infty} f(x_{k_i}) \leq f_\star + \mu\left((R_f+c)M_1 + \frac{1}{2}v(R_f+1)^2 + R_T\left(R_f+1+\frac{c}{2}\right)\right)^p + \epsilon,$$

(ii)
$$\liminf_{i\to\infty} \|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\|^2 \leq \frac{2}{\liminf_{i\to\infty}(1-\alpha_i)}\left(2v(R_f+1)M_2 + 2R_T M_2 + R_T^2\right),$$

where $M_1, M_2 \geq 0$ are constants whose existence is guaranteed by Lemmas 3.1 and 3.2.

Proof. Fix $x^\star \in X^\star$ arbitrarily. We will prove the assertion by separating the problem into two cases: the case where a number $k_0 \in \mathbb{N}$ exists such that $\|x_{k+1} - x^\star\| \leq \|x_k - x^\star\|$ for all $k \geq k_0$, and its negation.

(Positive case). First, let us consider the positive case, i.e., there is a number $k_0 \in \mathbb{N}$ such that $\|x_{k+1} - x^\star\| \leq \|x_k - x^\star\|$ for all $k \geq k_0$. Here, let us prove the existence of a subsequence that satisfies the property (i). We will proceed by way of contradiction, and suppose that

$$\liminf_{k\to\infty} f(x_k) > f_\star + \mu\left((R_f+c)M_1 + \frac{1}{2}v(R_f+1)^2 + R_T\left(R_f+1+\frac{c}{2}\right)\right)^p + \epsilon.$$

The strictness of the above inequality guarantees the existence of positive constants $\delta_1, \delta_2, \delta_3, \delta_4 > 0$ such that

$$\begin{aligned}\liminf_{k\to\infty} f(x_k) \geq f_\star + \mu\Bigg((R_f+c)M_1 &+ \frac{1}{2}(v+\delta_1)(R_f+1)^2 \\ &+ (R_T+\delta_2)\left(R_f+1+\frac{c}{2}\right) + \delta_3\Bigg)^p + \epsilon + 2\delta_4.\end{aligned}$$

Here, there exists a number $k_1 \in \mathbb{N}$ such that $v_k < v + \delta_1$ for all $k \geq k_1$ since the sequence v_k converges to the constant v. From Assumption (A7), the property of the limit superior guarantees the existence of a number $k_2 \in \mathbb{N}$ such that it is greater than k_1 and $\left\|r_k^T\right\| < R_T + \delta_2$ for all $k \geq k_2$. Similarly, there exists a number $k_3 \in \mathbb{N}$ such that it is greater than k_2 and $\epsilon_k < \epsilon + \delta_4$ for all $k \geq k_3$. Furthermore, the property of the limit inferior ensures the existence of a number $k_4 \geq k_3$ such that $f(x_k) > \liminf_{k\to\infty} f(x_k) - \delta_4$ for all $k \geq k_4$. Hence, we have

$$f(x_k) > f_\star + \mu\left((R_f+c)M_1 + \frac{1}{2}v_k(R_f+1)^2 + \left\|r_k^T\right\|\left(R_f+1+\frac{c}{2}\right) + \delta_3\right)^p + \epsilon_k.$$

for all $k \geq k_4$. Lemma 3.3 with the above inequality guarantees that

$$\|x_{k+1} - x^\star\|^2 \leq \|x_k - x^\star\|^2 - 2v_k\delta_3(1-\alpha_k).$$

for all $k \geq k_4$. Assumption (A6) implies that a number $k_5 \geq k_4$ exists such that $\alpha_k < (1 + \limsup_{k\to\infty} \alpha_k)/2$ for all $k \geq k_5$. Hence, we have

$$\begin{aligned} \|x_{k+1} - x^\star\|^2 &\leq \|x_k - x^\star\|^2 - v_k \delta_3 \left(1 - \limsup_{k\to\infty} \alpha_k\right) \\ &\leq \|x_{k_5} - x^\star\|^2 - \delta_3 \left(1 - \limsup_{k\to\infty} \alpha_k\right) \sum_{j=k_5}^{k} v_j \end{aligned}$$

for all $k \geq k_5$. Assumption (A6) guarantees that $\limsup_{k\to\infty} \alpha_k$ is strictly less than 1. Therefore, the assumption $\sum_{k=1}^{\infty} v_k = \infty$ means that the above inequality does not hold for large enough $k \geq k_5$, and we have arrived at a contradiction. Hence, we have

$$\liminf_{k\to\infty} f(x_k) \leq f_\star + \mu \left((R_f + c)M_1 + \frac{1}{2} v (R_f + 1)^2 + R_T \left(R_f + 1 + \frac{c}{2} \right) \right)^p + \epsilon.$$

Hence, the above inequality implies the existence of a subsequence $\{x_{k_i}\} \subset \{x_k\}$ which satisfies

$$\begin{aligned} \lim_{i\to\infty} f(x_{k_i}) &= \liminf_{k\to\infty} f(x_k) \\ &\leq f_\star + \mu \left((R_f + c)M_1 + \frac{1}{2} v (R_f + 1)^2 + R_T \left(R_f + 1 + \frac{c}{2} \right) \right)^p + \epsilon. \end{aligned}$$

Let us prove that this subsequence also satisfies the property (ii). From Lemma 3.2, we have

$$\begin{aligned} &(1 - \alpha_{k_i}) \|x_{k_i} - T(x_{k_i} - v_{k_i} \tilde{g}_{k_i})\|^2 + \|x_{k_i+1} - x^\star\|^2 \\ &\leq \|x_{k_i} - x^\star\|^2 + 2 v_{k_i} (R_f + 1) M_2 + 2 \left\| r_{k_i}^T \right\| M_2 + \left\| r_{k_i}^T \right\|^2. \end{aligned}$$

Here, there exists a beginning number $i_0 \in \mathbb{N}$ such that every following number $i \geq i_0$ satisfies that $\liminf_{j\to\infty}(1 - \alpha_j)/2 < 1 - \alpha_{k_i}$ because of the property of the positive limit inferior $\liminf_{j\to\infty}(1 - \alpha_j)$. Hence, we have

$$\begin{aligned} &\frac{1}{2} \left(\liminf_{j\to\infty} (1 - \alpha_j) \right) \|x_{k_i} - T(x_{k_i} - v_{k_i} \tilde{g}_{k_i})\|^2 + \|x_{k_i+1} - x^\star\|^2 \\ &\leq \|x_{k_i} - x^\star\|^2 + 2 v_{k_i} (R_f + 1) M_2 + 2 \left\| r_{k_i}^T \right\| M_2 + \left\| r_{k_i}^T \right\|^2 \\ &\leq \|x_{k_i} - x^\star\|^2 + 2 v_{k_i} (R_f + 1) M_2 + 2 \left(\sup_{k \geq k_i} \left\| r_k^T \right\| \right) M_2 + \left(\sup_{k \geq k_i} \left\| r_k^T \right\| \right)^2 \end{aligned}$$

for all $i \geq i_0$. Each term on the right side of the above inequality without the term $\|x_{k_i} - x^\star\|^2$ converges because the assumption of this theorem guarantees the convergence of the sequence $\{v_{k_i}\}$ and other terms are monotone decreasing and bounded from below with respect to the subscript $i \in \mathbb{N}$. Therefore, the fundamental properties on the inequality relationships of the limit inferior [12, Section 1.4] lead to

$$\frac{1}{2}\left(\liminf_{j\to\infty}(1-\alpha_j)\right)\liminf_{i\to\infty}\|x_{k_i}-T(x_{k_i}-v_{k_i}\tilde{g}_{k_i})\|^2+\liminf_{i\to\infty}\|x_{k_i+1}-x^\star\|^2$$
$$\leq \liminf_{i\to\infty}\left(\|x_{k_i}-x^\star\|^2+2v_{k_i}(R_f+1)M_2+2\left(\sup_{k\geq k_i}\left\|r_k^T\right\|\right)M_2+\left(\sup_{k\geq k_i}\left\|r_k^T\right\|\right)^2\right)$$
$$=\liminf_{i\to\infty}\|x_{k_i}-x^\star\|^2+2v(R_f+1)M_2+2R_TM_2+R_T^2.$$

Here, we have assumed that the sequence $\{\|x_k-x^\star\|\}$ is monotone decreasing in this case. Furthermore, this sequence is bounded from below. Hence, it converges. Therefore, the limit point of the sequence $\{\|x_k-x^\star\|\}$ is unique, i.e.,

$$\liminf_{i\to\infty}\|x_{k_i+1}-x^\star\|=\liminf_{i\to\infty}\|x_{k_i}-x^\star\|=\lim_{k\to\infty}\|x_k-x^\star\|.$$

This implies that

$$\frac{1}{2}\left(\liminf_{j\to\infty}(1-\alpha_j)\right)\liminf_{i\to\infty}\|x_{k_i}-T(x_{k_i}-v_{k_i}\tilde{g}_{k_i})\|^2$$
$$\leq 2v(R_f+1)M_2+2R_TM_2+R_T^2.$$

From Assumption (A6), $\liminf_{j\to\infty}(1-\alpha_j)$ is not zero. Hence, in the positive case, there exists a subsequence $\{x_{k_i}\}\subset\{x_k\}$ which satisfies properties (i) and (ii).

(Negative case). Next, let us consider the negative case, in other words, the case where a subsequence $\{x_{k_i}\}\subset\{x_k\}$ exists that satisfies $\|x_{k_i}-x^\star\|<\|x_{k_i+1}-x^\star\|$ for all $i\in\mathbb{N}$. Similarly to the positive case, let us prove that the subsequence $\{x_{k_i}\}$ satisfies the property (i). Fix $i\in\mathbb{N}$ arbitrarily. We will proceed by way of contradiction, and suppose that

$$f(x_{k_i})>f_\star+\mu\left((R_f+c)M_1+\frac{1}{2}v_{k_i}(R_f+1)^2+\left\|r_{k_i}^T\right\|\left(R_f+1+\frac{c}{2}\right)+\frac{1}{i}\right)^p+\epsilon_{k_i}.$$

From Lemma 3.3, we have

$$\|x_{k_i+1}-x^\star\|^2\leq\|x_{k_i}-x^\star\|^2-\frac{2v_{k_i}}{i}(1-\alpha_{k_i}).$$

The assumption in this case, $\|x_{k_i}-x^\star\|<\|x_{k_i+1}-x^\star\|$, contradicts the above inequality. Hence, we have

$$f(x_{k_i})$$
$$\leq f_\star+\mu\left((R_f+c)M_1+\frac{1}{2}v_{k_i}(R_f+1)^2+\left\|r_{k_i}^T\right\|\left(R_f+1+\frac{c}{2}\right)+\frac{1}{i}\right)^p+\epsilon_{k_i}$$
$$\leq f_\star+\mu\left((R_f+c)M_1+\frac{1}{2}v_{k_i}(R_f+1)^2+\left(\sup_{k\geq k_i}\left\|r_k^T\right\|\right)\left(R_f+1+\frac{c}{2}\right)+\frac{1}{i}\right)^p+\sup_{k\geq k_i}\epsilon_k$$

for all $i \in \mathbb{N}$. This implies that

$$\liminf_{i\to\infty} f(x_{k_i}) \le f_\star + \mu\left((R_f+c)M_1 + \frac{1}{2}v(R_f+1)^2 + R_k^T\left(R_f+1+\frac{c}{2}\right)\right)^p + \epsilon,$$

and thus, a subsequence $\{x_{k_{i_j}}\} \subset \{x_{k_i}\}$ exists that satisfies property (i). Let us prove that this subsequence also satisfies the property (ii). From Lemma 3.2, we have

$$\begin{aligned}
&(1-\alpha_{k_{i_j}})\left\|x_{k_{i_j}} - T(x_{k_{i_j}} - v_{k_{i_j}}\tilde{g}_{k_{i_j}})\right\|^2 + \left\|x_{k_{i_j}+1} - x^\star\right\|^2 \\
&\le \left\|x_{k_{i_j}} - x^\star\right\|^2 + 2v_{k_{i_j}}(R_f+1)M_2 + 2\left\|r^T_{k_{i_j}}\right\|M_2 + \left\|r^T_{k_{i_j}}\right\|^2.
\end{aligned}$$

Here, there exists a beginning number $j_0 \in \mathbb{N}$ such that every following number $j \ge j_0$ satisfies that $\liminf_{j\to\infty}(1-\alpha_j)/2 < 1-\alpha_{k_{i_j}}$ because of the property of the positive limit inferior $\liminf_{j\to\infty}(1-\alpha_j)$. Hence, we have

$$\begin{aligned}
&\frac{1}{2}\left(\liminf_{l\to\infty}(1-\alpha_l)\right)\left\|x_{k_{i_j}} - T(x_{k_{i_j}} - v_{k_{i_j}}\tilde{g}_{k_{i_j}})\right\|^2 + \left\|x_{k_{i_j}+1} - x^\star\right\|^2 \\
&\le \left\|x_{k_{i_j}} - x^\star\right\|^2 + 2v_{k_{i_j}}(R_f+1)M_2 + 2\left\|r^T_{k_{i_j}}\right\|M_2 + \left\|r^T_{k_{i_j}}\right\|^2 \\
&\le \left\|x_{k_{i_j}} - x^\star\right\|^2 + 2v_{k_{i_j}}(R_f+1)M_2 + 2\left(\sup_{k\ge k_{i_j}}\left\|r_k^T\right\|\right)M_2 + \left(\sup_{k\ge k_{i_j}}\left\|r_k^T\right\|\right)^2
\end{aligned}$$

for all $j \ge j_0$. Each term on the right side of the above inequality without the term $\left\|x_{k_{i_j}} - x^\star\right\|^2$ converges, as we saw in the positive case. Hence, we have

$$\begin{aligned}
&\frac{1}{2}\left(\liminf_{l\to\infty}(1-\alpha_l)\right)\liminf_{j\to\infty}\left\|x_{k_{i_j}} - T(x_{k_{i_j}} - v_{k_{i_j}}\tilde{g}_{k_{i_j}})\right\|^2 + \liminf_{j\to\infty}\left\|x_{k_{i_j}+1} - x^\star\right\|^2 \\
&\le \liminf_{j\to\infty}\left\|x_{k_{i_j}} - x^\star\right\|^2 + 2v(R_f+1)M_2 + 2R_TM_2 + R_T^2.
\end{aligned}$$

Here, the assumption in this case implies

$$\liminf_{j\to\infty}\left\|x_{k_{i_j}} - x^\star\right\|^2 < \liminf_{j\to\infty}\left\|x_{k_{i_j}+1} - x^\star\right\|^2.$$

Hence, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\liminf_{l\to\infty}(1-\alpha_l)\right)\liminf_{j\to\infty}\left\|x_{k_{i_j}} - T(x_{k_{i_j}} - v_{k_{i_j}}\tilde{g}_{k_{i_j}})\right\|^2 \\
&\le 2v(R_f+1)M_2 + 2R_TM_2 + R_T^2.
\end{aligned}$$

From Assumption (A6), $\liminf_{j\to\infty}(1-\alpha_j)$ is not zero. Therefore, the subsequence $\{x_{k_{i_j}}\} \subset \{x_k\}$ satisfies properties (i) and (ii). This completes the proof. □

The following theorem is the main theorem of this paper. This theorem extends the existing theorems [4, Theorems 3.1 and 3.2] of the exact fixed point quasiconvex subgradient method for both constant and diminishing step-size rules. We can obtain them by letting ϵ_k, r_k^f, r_k^T and c be zero for all $k \in \mathbb{N}$.

Theorem 3.5. *Let $\{x_k\}$ be the sequence generated by Algorithm 1, and let Assumption 2.1 hold. Then, there exists a subsequence $\{x_{k_i}\} \subset \{x_k\}$ which satisfies*

(i) $$\lim_{i\to\infty} f(x_{k_i}) \leq f_\star + \mu\left((R_f + c)M_{1,2,\alpha} + \frac{1}{2}v(R_f+1)^2 + R_T\left(R_f + 1 + \frac{c}{2}\right)\right)^p + \epsilon,$$

(ii) $$\lim_{i\to\infty} \|x_{k_i} - T(x_{k_i})\|^2 \leq 2v(R_f+1)M_{1,2,\alpha} + 4v^2(R_f+1)^2 + R_T(R_T + M_{1,2,\alpha})$$

for some constant $M_{1,2,\alpha} \geq 0$ which is determined by M_1 in Lemma 3.1, M_2 in Lemma 3.2, and the sequence $\{\alpha_k\}$.

Proof. Lemma 3.4 ensures the existence of a subsequence $\{x_{k_i}\} \subset \{x_k\}$ which satisfies

$$\lim_{i\to\infty} f(x_{k_i}) \leq f_\star + \mu\left((R_f + c)M_1 + \frac{1}{2}v(R_f+1)^2 + R_T\left(R_f + 1 + \frac{c}{2}\right)\right)^p + \epsilon, \tag{3.6}$$

and

$$\liminf_{i\to\infty} \|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\|^2 \leq \frac{2}{\liminf_{i\to\infty}(1-\alpha_i)}\left(2v(R_f+1)M_2 + 2R_T M_2 + R_T^2\right). \tag{3.7}$$

In the following discussion, let us consider this subsequence. Fix $i \in \mathbb{N}$ arbitrarily. $\|x_{k_i} - T(x_{k_i})\|^2$ can be expanded with the triangle inequality by noting the nonexpansivity of T as follows:

(3.8)
$$\begin{aligned}
&\|x_{k_i} - T(x_{k_i})\|^2 \\
&\leq \left(\|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\| + \|T(x_{k_i} - v_{k_i}\tilde{g}_{k_i}) - T(x_{k_i})\|\right)^2 \\
&\leq \left(\|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\| + v_{k_i}\|\tilde{g}_{k_i}\|\right)^2 \\
&= \|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\|^2 + 2v_{k_i}\|\tilde{g}_{k_i}\|\,\|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\| + v_{k_i}^2\|\tilde{g}_{k_i}\|^2 \\
&\leq \|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\|^2 + 2v_{k_i}(R_f+1)\,\|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\| + v_{k_i}^2(R_f+1)^2.
\end{aligned}$$

Here, an upper bound of the term $\|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\|$ can be estimated, from the nonexpansivity of T with an arbitrarily chosen fixed point $x^\star \in$

$X^\star (\neq \emptyset)$, at

$$\begin{aligned}\|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\| &\leq \|x_{k_i} - x^\star\| + \|x^\star - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\| \\ &\leq 2\,\|x_{k_i} - x^\star\| + v_{k_i}\,\|\tilde{g}_{k_i}\|\end{aligned}$$

for all $i \in \mathbb{N}$. Here, the sequence $\{\|x_{k_i} - x^\star\|\} \subset \{\|x_k - x^\star\|\}$ is bounded from above by a constant M_1 whose existence is guaranteed by Lemma 3.1, and the sequence $\{\|\tilde{g}_{k_i}\|\} \subset \{\|\tilde{g}_k\|\}$ is bounded from above by a constant $R_f + 1$, as we saw in the proof of Lemma 3.1. Hence, we obtain

$$\|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\| \leq 2M_1 + v_{k_i}(R_f + 1)$$

for all $i \in \mathbb{N}$. Hence, using inequality (3.8), we have

$$\begin{aligned}&\|x_{k_i} - T(x_{k_i})\|^2 \\ &\leq \|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\|^2 + 2v_{k_i}(R_f + 1)(2M_1 + v_{k_i}(R_f + 1)) + v_{k_i}^2(R_f + 1)^2 \\ &\leq \|x_{k_i} - T(x_{k_i} - v_{k_i}\tilde{g}_{k_i})\|^2 + 4v_{k_i}(R_f + 1)(M_1 + v_{k_i}(R_f + 1))\end{aligned}$$

for all $i \in \mathbb{N}$. Together with the convergence of the sequence $\{v_k\}$ and inequality (3.7), we have

$$\begin{aligned}&\liminf_{i\to\infty} \|x_{k_i} - T(x_{k_i})\|^2 \\ &\leq \liminf_{i\to\infty} \|x_{k_i} - T(x_{k_i} - v\tilde{g}_{k_i})\|^2 + 4v(R_f + 1)(M_1 + v(R_f + 1)) \\ &\leq \frac{2}{\liminf_{j\to\infty}(1 - \alpha_j)} \left(2v(R_f + 1)M_2 + 2R_T M_2 + R_T^2\right) \\ &\quad + 4v(R_f + 1)(M_1 + v(R_f + 1)).\end{aligned}$$

Here, let us define $M_{1,2,\alpha} := \max\{4M_1, 4M_2/\liminf_{k\to\infty}(1 - \alpha_{k_i})\} < \infty$. Accordingly, we have the desired inequality:

$$\liminf_{k\to\infty} \|x_{k_i} - T(x_{k_i})\|^2 \leq 2v(R_f + 1)M_{1,2,\alpha} + 4v^2(R_f + 1)^2 + R_T(R_T + M_{1,2,\alpha}).$$

This completes the proof. □

4. Conclusion

We discussed the convergence of the fixed point quasiconvex subgradient method in the case where some inexactness exists on the computation. The main theorem extends the existing results for the exact fixed point quasiconvex subgradient method and reveals the convergence property of the fixed point quasiconvex subgradient method when computational inexactness exists.

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